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# Spectral and formal stability criteria of spatially inhomogeneous stationary solutions to the Vlasov equation for the Hamiltonian mean-field model

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Stability of spatially inhomogeneous solutions to the Vlasov equation is investigated for the Hamiltonian mean-field model to provide the spectral and formal stability criteria in the form of necessary and sufficient conditions. These criteria determine stability of spatially inhomogeneous solutions whose stability has not been decided correctly by using a less refined formal stability criterion. It is shown that some of such solutions can be found in a family of stationary solutions to the Vlasov equation, which is parametrized with macroscopic quantities and has a two-phase coexistence region in the parameter space.

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## I. INTRODUCTION

The macroscopic behavior of many-body systems depends on whether the interaction is of long or short range. For many-body systems with short-range interaction, the thermodynamic observables such as entropy and magnetization are additive and extensive but not so for those with long-range interaction. The additivity and the extensivity of observables are assumed to hold in the equilibrium statistical mechanics and thermodynamics. Macroscopic behaviors of many-body systems with long-range interaction are quite different from those with a short-range one [1–4]. The long-range interaction system is likely to be trapped in quasistationary states (QSSs), and accordingly a very long time is needed to reach the thermal equilibrium state. The duration of those QSSs increases according to the system size, and diverges if one takes the thermodynamic limit [5–11]. It is a widely accepted understanding that the equilibration is brought about by the finite size effect.

A way to analyze a Hamiltonian system with long-range interaction is to use the Vlasov equation or collisionless Boltzmann equation [5,12], which can be derived by taking the limit of  $N \rightarrow \infty$ , where  $N$  is the number of elements [13–15]. The QSSs are supposed to be associated with stable stationary solutions to the Vlasov equation [2,5]. Finding a stability criterion for stationary solutions to the Vlasov equation is the first step to investigate QSSs since such a criterion makes it possible to decide whether a stationary solution can be a QSS or not.

The stability of solutions to the Vlasov equation has been investigated in [5–7,16–24]. There are several concepts of stability such as the spectral stability, the linear stability, the formal stability, and the nonlinear stability [16]. The interest of this paper centers on the spectral stability and the formal stability, but the linear stability and the nonlinear stability are not touched upon. The formal [6,16] and spectral [17,21,22] stability criteria for spatially homogeneous solutions have been well known already.

Meanwhile, the stability of spatially inhomogeneous solutions has been investigated in the astrophysics [5,18–20] since around a half century ago. The stability for the spherical

galaxy is rigorously investigated recently [24]. Antonov's variational principle [5,18] particularly gives a necessary and sufficient condition for stability of some stationary solution by considering stability against not all perturbations but only accessible perturbations called phase preserving perturbations [20]. The restriction for the perturbations comes from the fact that the Vlasov equation has an infinite number of invariants. We note that the stability of a given stationary state can not be determined by using the stability criterion given in a statement of Antonov's variational principle [5] practically.

In the context of statistical physics for QSSs, the stability of spatially inhomogeneous solutions to the Vlasov equation has been studied, say, by Campa and Chavanis [23]. They set up criteria for formal stability both in the most refined form and in less refined forms by using the fact that accessible perturbations conserve all Casimir invariants at linear order. We call the most refined formal stability, simply, the formal stability in this paper. Their formal stability criterion in the most refined form requires one to take into account an infinite number of Casimir invariants and to detect an infinite number of associated Lagrangian multipliers in order to determine the stability of spatially inhomogeneous stationary solutions. Their formal stability criterion is hence hard to use. In contrast with this, the canonical formal stability criterion which is one of the less refined formal stability criteria is of practical use. Using the canonical formal stability criterion, one can check stability of a stationary state against a perturbation which keeps the normalization condition but may break the energy conservation and other Casimir invariant conditions. Although the criterion for canonical formal stability is stated as a necessary and sufficient condition, it is just a sufficient condition for the formal stability. It is to be expected that a criterion for the formal stability is found out in the form of necessary and sufficient condition without reference to an infinite number of quantities such as Lagrangian multipliers.

This article deals with the Hamiltonian mean-field (HMF) model [25,26] with the anticipation stated above. The HMF model is a simple toy model which shows typical long-range features. For instance, the HMF model has been used for investigating the nonequilibrium phase transitions [27–30], the core-halo structure [31], the creation of small traveling clusters [32], the construction of traveling clusters [33], and a relaxation process with long-range interactions [6,7]. Moreover, the HMF model allows one to perform theoretical

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study on dynamics near spatially inhomogeneous stationary solutions to the Vlasov equation by the use of the dispersion function which can be explicitly written out for the HMF model. For instance, the dynamics of a perturbation around the spatially inhomogeneous stationary solution [34] and the algebraic damping to a QSS [35] have been investigated theoretically and numerically by using the HMF model. Further, the linear response to the external field is studied in an explicit form [36] for a spatially inhomogeneous QSS. In those studies, the stability of the spatially inhomogeneous solutions have been assumed to hold, and then it is worthwhile to give an explicit form of necessary and sufficient condition for the stability of the spatially inhomogeneous stationary solutions. The aim of this article is to find spectral and formal stability criteria for spatially inhomogeneous stationary solutions. The spectral stability criterion is derived by means of the dispersion relation. The formal stability criterion is obtained by using the same idea as exhibited in [23]. The criterion we are to find by using the angle-action variables is free from an infinite number of Lagrangian multipliers, and is stated in the form of a necessary and sufficient condition, which allows us to look into the stability of spatially inhomogeneous solutions in an accessible manner.

This article is organized as follows. Section II contains a brief review of the two kinds of stabilities of a fixed point of a dynamical system. The nonlinear and linearized Vlasov equations for the HMF model are introduced in Sec. III. The spectral stability criterion for spatially homogeneous solutions to the Vlasov equation is given in Sec. IV in a rather simple method than that already known. By using the same method, the spectral stability criterion for spatially inhomogeneous solutions is obtained in Sec. VB. The formal stability criterion for spatially inhomogeneous solutions is derived in Sec. VD. In Sec. VE, we look into stability of a spatially inhomogeneous water-bag distribution by using the obtained criterion. Section VI gives an example which shows that the present stability criterion is of great use. It is shown that there is a family of stationary solutions whose stability can not be judged correctly by using the canonical formal stability criterion, but can be done by the criterion given in this article. Section VII is devoted to a summary and a discussion for generalization.

## II. SPECTRAL STABILITY AND FORMAL STABILITY

We start with a brief review of definitions of spectral stability and formal stability, following Holm *et al.* [16]. Let  $X$  be a normed space. Suppose that a dynamical system is given by the equation

$$\frac{dx}{dt} = f(x), \quad x \in X. \quad (1)$$

Let  $x_*$  be a fixed point of this system  $f(x_*) = 0$ . Then, the linearized equation around  $x_*$  is expressed as

$$\frac{d\xi}{dt} = Df(x_*)[\xi], \quad (2)$$

where  $Df(x_*)$  is a linear operator derived from  $f$  at  $x_*$ . The spectral stability and the formal stability of the fixed point  $x_*$  are defined as follows:

(i) The fixed point  $x_*$  is said to be *spectrally stable* if the linear operator  $Df(x_*)$  has no spectrum with positive real part. In addition, if the linear operator  $Df(x_*)$  has an eigenvalue with vanishing real part,  $x_*$  is called *neutrally spectrally stable*. The fixed point  $x_*$  is said to be spectrally unstable when there exists a spectrum with positive real part.

(ii) The fixed point  $x_*$  is said to be *formally stable* if a conserved functional  $\mathcal{F}[x]$  takes a critical value at  $x = x_*$  and further the second variation of  $\mathcal{F}$  at  $x_*$  is negative (or positive) definite. The fixed point  $x_*$  is said to be neutrally formally stable if the second variation of  $\mathcal{F}$  at  $x_*$  is negative (resp. positive) semidefinite but not negative (resp. positive) definite. Further, the fixed point  $x_*$  is said to be formally unstable if the second variation of  $\mathcal{F}$  at  $x_*$  is not negative (or positive) semidefinite.

We note that the formal stability can be defined for  $x^*$  which is a critical point of  $\mathcal{F}$  under some constraints coming from invariants of the dynamical system in question.

If the dynamical system in question is infinite dimensional, the fixed point  $x_*$  is occasionally called a stationary state. We note that the definition of neutral spectral stability is different from the original one in [16]. The detail of our footing for stability analysis is exhibited in Appendix A. According to [16], the neutrally spectrally stable solution is spectrally stable, but the neutrally formally stable solution is not formally stable.

## III. VLASOV EQUATION FOR HAMILTONIAN MEAN-FIELD MODEL

The Hamiltonian mean-field (HMF) model [25,26] for  $N$  unit mass particles on the unit circle  $S^1$  has the Hamiltonian given by

$$H_N = \sum_{i=1}^N \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^N [1 - \cos(q_i - q_j)], \quad (3)$$

$$p_i \in \mathbb{R}, \quad q_i \in [-\pi, \pi), \quad i = 1, 2, \dots, N.$$

In the limit of  $N$  tending to infinity, the time evolution of the HMF model can be described in terms of a single-body distribution  $f$  on the  $\mu$  space which coincides with  $S^1 \times \mathbb{R}$ . The single-body distribution  $f$  is known to evolve according to the Vlasov equation

$$\frac{\partial f}{\partial t} + \{\mathcal{H}[f], f\} = 0, \quad (4)$$

where  $\mathcal{H}[f]$  is the effective single-body Hamiltonian defined to be

$$\mathcal{H}[f] = \frac{p^2}{2} + \mathcal{V}[f](q, t),$$

$$\mathcal{V}[f] = - \int_{-\pi}^{\pi} dq' \cos(q - q') \int_{-\infty}^{\infty} f(q', p', t) dp', \quad (5)$$

$$p \in \mathbb{R}, \quad q \in [-\pi, \pi),$$

and where  $\{a, b\}$  is the Poisson bracket given by

$$\{a, b\} = \frac{\partial a}{\partial p} \frac{\partial b}{\partial q} - \frac{\partial a}{\partial q} \frac{\partial b}{\partial p}. \quad (6)$$

By averaging the distribution  $f$  in the  $x$  and  $y$  directions, the order parameter  $\vec{M}[f] = (M_x[f], M_y[f])^T$  is defined to be

$$\begin{aligned} M_x[f](t) &= \iint_{\mu} \cos q f(q, p, t) dq dp, \\ M_y[f](t) &= \iint_{\mu} \sin q f(q, p, t) dq dp, \end{aligned} \quad (7)$$

where the symbol  $\mu$  denotes the whole  $\mu$  space  $S^1 \times \mathbb{R}$ . In the Vlasov dynamics, a functional

$$Q[f] \equiv \iint_{\mu} Q(f(q, p, t)) dq dp \quad (8)$$

is conserved for any function  $Q$ , and such a functional is called a Casimir invariant.

Let  $f_0$  denote a stationary solution to the Vlasov equation. Then, a small perturbation  $f_1$  around  $f_0$  is shown to obey, in some time scale, the linearized Vlasov equation

$$\frac{\partial f_1}{\partial t} = \hat{L} f_1, \quad \hat{L} f_1 := -\{\mathcal{H}[f_0], f_1\} - \{\mathcal{V}[f_1], f_0\}. \quad (9)$$

This equation can be analyzed by means of the Fourier-Laplace transformation. For the sake of physical interpretation, we define the Laplace transform of a function  $g(t)$  to be

$$\tilde{g}(\omega) = \int_0^{\infty} g(t) e^{i\omega t} dt, \quad \text{Im} \omega > 0. \quad (10)$$

Through the Fourier series expansion with respect to  $q$  and the Laplace transformation with respect to  $t$ , Eq. (9) is brought into the dispersion relation  $D(\omega) = 0$  [12,37]. The explicit form of the dispersion relations for spatially homogeneous stationary states and for spatially inhomogeneous stationary states will be exhibited in Secs. IV and V, respectively. We call  $D(\omega)$  a dispersion function, which is said to be a dielectric function in the context of the plasmas physics. A root  $\omega$  of this dispersion relation with positive imaginary part is in one-to-one correspondence with the eigenvalue  $-i\omega$  of the linear operator  $\hat{L}$ . The detail of this fact is reviewed in Appendix B. It then follows that the stationary solution  $f_0$  is spectrally unstable if the dispersion relation  $D(\omega) = 0$  has a root in the upper half  $\omega$  plane.

Although the domain of the dispersion function  $D(\omega)$  is the upper half  $\omega$  plane, it can be analytically continued to the lower half  $\omega$  plane [12,37]. The root of the dispersion relation on the lower half plane causes the exponential damping of the order parameter for the small perturbation  $\vec{M}[f_1] = (M_x[f_1], M_y[f_1])^T$ , which is called the Landau damping [12,37,38]. Such a root is not an eigenvalue, but is called a resonance pole, a Landau pole, or a fake eigenvalue.

We remark on embedded eigenvalues of the linear operator  $\hat{L}$  on the imaginary axis. The linear operator  $\hat{L}$  may have continuous spectra lying on the imaginary axis [39]. An “eigenvalue” with zero real part is occasionally embedded in continuous spectra and such an “eigenvalue” is called an embedded eigenvalue.

Let a stationary solution  $f_0$  be spatially inhomogeneous, i.e.,  $|\vec{M}[f_0]| \neq 0$ . Owing to the rotational symmetry of the HMF model, there must be a mode  $f_1^0$  associated with the zero embedded eigenvalue of the linearized Vlasov operator. The direction of a perturbation  $\vec{M}[f_1^0]$  is perpendicular to the

direction of the order parameter  $\vec{M}[f_0]$ . This fact is consistent with the fact that the order parameter changes its direction, keeping its radius, in the presence of an infinitesimal external field which is perpendicular to the order parameter [36]. When the nonequilibrium phase transition [27–30] is of interest, the phases are defined by the modulus of the order parameter, and hence the direction of the order parameter is not questioned. At present, we mainly focus on the stability against the perturbation parallel to the order parameter  $\vec{M}[f_0]$ .

#### IV. STABILITY CRITERION FOR SPATIALLY HOMOGENEOUS STATIONARY SOLUTION

As long as the linear operator  $\hat{L}$  defined in (9) is concerned, the spectral stability condition for a stationary solution  $f_0$  to the Vlasov equation can be compactly stated; if there is no eigenvalue of  $\hat{L}$ , the stationary solution  $f_0$  is said to be spectrally stable. Since the spectrum of  $\hat{L}$  consists of eigenvalues on  $\mathbb{C} \setminus i\mathbb{R}$ , continuous spectra on the imaginary axis, and the embedded eigenvalue on the imaginary axis [39], only the eigenvalues are able to contribute to the spectral instability. On account of this fact, the spectral stability criterion is stated as follows:

*Proposition 1.* Let  $f_0(p)$  be a spatially homogeneous stationary solution to the Vlasov equation, which is assumed to be smooth, even, and unimodal, and further the derivative  $f'_0(p)$  of which is assumed to have the support  $\mathbb{R}$ . Then the stationary solution  $f_0$  is spectrally stable, if and only if  $f_0$  satisfies the inequality

$$I[f_0] = 1 + \pi \int_{-\infty}^{\infty} \frac{f'_0(p)}{p} dp \geq 0. \quad (11)$$

We note that  $f'_0(p)/p$  has no singularity for all  $p \in \mathbb{R}$  on account of the assumption that  $f_0$  is smooth and even. Although the inequality (11) can be derived by using the Nyquist’s method [17,21,22], we introduce a method other than the Nyquist’s method to prove this proposition.

For a spatially homogeneous stationary solution  $f_0(p)$ , the dispersion relation  $D(\omega) = 0$  with  $\text{Im} \omega > 0$  is put in the form [2]

$$D(\omega) = 1 + \pi \int_{-\infty}^{\infty} \frac{f'_0(p)}{p - \omega} dp = 0, \quad \text{Im} \omega > 0. \quad (12)$$

The dispersion function is continued to  $\omega = 0$  from the upper half  $\omega$  plane, by taking the limit  $D(0) = \lim_{\epsilon \rightarrow 0+} D(i\epsilon)$ . Noting that  $f'_0(p)/p$  has no singularity, we obtain  $D(0) = I[f_0]$  since the integrand in (12) has no singularity when  $\omega = 0$ . We put  $\omega \in \mathbb{C}$  in the form  $\omega = \omega_r + i\omega_i$  with  $\omega_r \in \mathbb{R}$  and  $\omega_i > 0$ . When the dispersion relation (12) is satisfied by some  $\omega$  with  $\text{Im} \omega > 0$ , the imaginary part of  $D(\omega)$  is zero, so that one has

$$\begin{aligned} \text{Im} D(\omega) &= \pi \omega_i \int_{-\infty}^{\infty} \frac{f'_0(p)}{(p - \omega_r)^2 + \omega_i^2} dp \\ &= 4\pi \omega_i \omega_r \int_0^{\infty} \frac{p f'_0(p) dp}{((p - \omega_r)^2 + \omega_i^2)((p + \omega_r)^2 + \omega_i^2)} \\ &= 0. \end{aligned} \quad (13)$$

Since  $f_0(p)$  is an even unimodal function,  $p f'_0(p) < 0$  for all  $p > 0$ . The integral in (13) is to be negative value, and (13)



implies that  $\omega_r = 0$  since  $\omega_i > 0$ . Conversely, if  $\omega_r = 0$ , the equality  $\text{Im}D(\omega) = 0$  holds true. The condition  $\omega_r = 0$  is then equivalent to the condition (13).

Now, on account of the fact  $pf'_0(p)$  is negative for all  $p \neq 0$ , and the dispersion function satisfies the inequality

$$D(i\omega_i) = 1 + \pi \int_{-\infty}^{\infty} \frac{pf'_0(p)}{p^2 + \omega_i^2} dp \geq I[f_0] \quad (14)$$

for all  $\omega_i \geq 0$ , where the equality is satisfied if and only if  $\omega_i = 0$ . This is because  $D(i\omega_i)$  becomes  $I[f_0]$  for  $\omega_i = 0$  and because  $D(i\omega_i)$  is a continuous and strictly increasing function with respect to  $\omega_i$ . It then follows that if  $D(\omega) = 0$  with  $\omega = i\omega_i$ , or equivalently, if  $\hat{L}$  has an eigenvalue  $-i\omega$  with  $\text{Im}\omega > 0$ , then  $I[f_0] < 0$ . Conversely, if the inequality  $I[f_0] < 0$  is satisfied, there exists a positive  $\omega_i$  such that  $D(i\omega_i) = 0$ . In fact,  $D(i\omega_i)$  is strictly increasing in  $\omega_i$  with  $D(0) = I[f_0]$  and  $D(i\omega_i) \rightarrow 1$  as  $\omega_i \rightarrow \infty$ . This means that  $\hat{L}$  has an unstable eigenvalue if and only if  $I[f_0] < 0$ . Thus, we have proved the spectral stability criterion (11).

In comparison with the spectral stability criterion (11), the formal stability criterion [6] is given by

$$I[f_0] > 0. \quad (15)$$

This inequality means that  $f_0$  is spectrally stable but not neutrally spectrally stable. This is because if  $D(0) = I[f_0] = 0$ , the linear operator  $\hat{L}$  has an embedded eigenvalue 0, and hence  $f_0$  is neutrally spectrally stable.

## V. STABILITY CRITERIA FOR SPATIALLY INHOMOGENEOUS STATIONARY SOLUTION

In this section, we will give necessary and sufficient conditions for the spectral stability and for the most refined formal stability of spatially inhomogeneous stationary solutions to the Vlasov equation. We call the most refined formal stability, simply, the formal stability as we have already mentioned in the Introduction.

A spectral stability criterion for spatially inhomogeneous solutions can be given in an explicit form by performing the same procedure as that adopted in the last section. Furthermore, the formal stability criterion can be worked out if all the Casimir invariants are taken into account. For spatially inhomogeneous stationary solutions, Campa and Chavanis [23] have given the formal stability criterion. However, no one has these criteria explicitly since one needs to detect values of an infinite number of Lagrangian multipliers. We can avoid a puzzle to detect an infinite number of Lagrangian multipliers if we use the angle-action coordinates in stability analysis.

We denote the single-body energy by

$$\mathcal{E}(q, p) = p^2/2 - M_0 \cos q, \quad (16)$$

where on account of the rotational symmetry of the HMF model, the order parameter has been set  $\vec{M}_0 = (M_0, 0)$  with

$$M_0 = \iint_{\mu} \cos q f_0(q, p) dq dp. \quad (17)$$

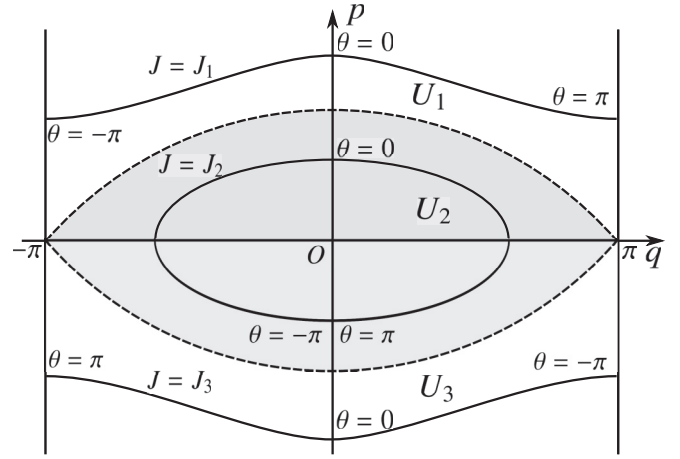


FIG. 1. We illustrate the angle-action variables in regions  $U_i$  ( $i = 1, 2, 3$ ) in the  $\mu$  space. The broken curve is a separatrix. The region  $U_2$  is the gray region surrounded by the separatrix. The solid curves in each region are trajectories of the dynamics induced by the effective single-body Hamiltonian  $\mathcal{H}[f_0](q, p) = p^2/2 - M_0 \cos q$ .

### A. Angle-action coordinates for HMF model

Before analyzing the stability of spatially inhomogeneous stationary solutions, we review the angle-action coordinates  $(\theta, J)$  for the HMF model. The detail of constructing the angle-action coordinates can be found in [34].

To construct a bijective mapping of  $(q, p)$  to  $(\theta, J)$ , we divide the  $\mu$  space into three regions,  $U_1, U_2$ , and  $U_3$ , which are defined, respectively, as

$$\begin{aligned} U_1 &= \{(q, p) | \mathcal{E}(q, p) > M_0, p > 0\}, \\ U_2 &= \{(q, p) | |\mathcal{E}(q, p)| < M_0\}, \\ U_3 &= \{(q, p) | \mathcal{E}(q, p) > M_0, p < 0\}. \end{aligned} \quad (18)$$

According to this division of the  $\mu$  space, we prepare the sets  $V_1, V_2$ , and  $V_3$  defined to be

$$\begin{aligned} V_1 &= \{(\theta_1, J_1) | \theta_1 \in [-\pi, \pi), J_1 > 4\sqrt{M_0}/\pi\}, \\ V_2 &= \{(\theta_2, J_2) | \theta_2 \in [-\pi, \pi), 0 < J_2 < 8\sqrt{M_0}/\pi\}, \\ V_3 &= \{(\theta_3, J_3) | \theta_3 \in [-\pi, \pi), J_3 > 4\sqrt{M_0}/\pi\}, \end{aligned} \quad (19)$$

respectively. Then, the maps  $(q, p) \mapsto (\theta_i, J_i) : U_i \rightarrow V_i$  for  $i = 1, 2, 3$ , are bijective. We illustrate the angle-action variables in three regions  $U_1, U_2$ , and  $U_3$  in Fig. 1. Since we are interested in integration over the  $\mu$  space, we do not have to mention more on the boundaries of  $U_i$ .

According to these bijections, a function  $g$  whose arguments are the angle-action variables  $(\theta, J)$  is denoted by

$$g(\theta, J) = \begin{cases} g_1(\theta_1, J_1), & (\theta, J) \in V_1 \\ g_2(\theta_2, J_2), & (\theta, J) \in V_2 \\ g_3(\theta_3, J_3), & (\theta, J) \in V_3 \end{cases} \quad (20)$$

respectively. We will omit the subscript  $i$  if no confusion arises. For notational simplicity, we denote the integral of the function (20) over the whole  $\mu$  space by the left-hand side of the following equation:

$$\iint_{\mu} g(\theta, J) d\theta dJ \equiv \sum_{i=1,2,3} \iint_{V_i} g_i(\theta_i, J_i) d\theta_i dJ_i. \quad (21)$$

In a similar manner, the integration of a function  $f(J)$  is put in the form

$$\int_L f(J) dJ \equiv \int_{4\sqrt{M_0}/\pi}^{\infty} f_1(J_1) dJ_1 + \int_0^{8\sqrt{M_0}/\pi} f_2(J_2) dJ_2 + \int_{4\sqrt{M_0}/\pi}^{\infty} f_3(J_3) dJ_3. \quad (22)$$

In the latter part of this article, the monotonicity of a function  $f(J)$  with respect to  $J$  means the monotonicity of functions  $f_i(J_i)$  with respect to  $J_i$  for each  $i = 1, 2, 3$ , respectively.

## B. Spectral stability criterion

We derive a necessary and sufficient condition for a spatially inhomogeneous stationary solution  $f_0$  to the Vlasov equation to be spectrally stable, which is stated as follows:

*Proposition 2.* Let  $f_0$  be a spatially inhomogeneous stationary solution to the Vlasov equation, which is assumed to depend on the action  $J$  only through the single-body energy  $\mathcal{E}(J)$  in such a manner that

$$f_0(q, p) = \tilde{f}_0(J(q, p)) = \hat{f}_0(\mathcal{E}(q, p)). \quad (23)$$

Further,  $\tilde{f}_0(J)$  and  $\hat{f}_0(\mathcal{E})$  are assumed to be strictly decreasing with respect to  $J$  and  $\mathcal{E}$ , respectively. A further assumption is that  $d\hat{f}_0(\mathcal{E})/d\mathcal{E}$  is continuous with respect to  $\mathcal{E}$ . Such a stationary solution  $f_0(q, p)$  is spectrally stable, if and only if

$$I[f_0] = 1 + \int_{-\pi}^{\pi} dq \cos^2 q \int_{-\infty}^{\infty} \frac{1}{p} \frac{\partial f_0}{\partial p}(q, p) dp - 2\pi \int_L \frac{\tilde{f}_0'(J)}{\Omega(J)} |C^0(J)|^2 dJ \geq 0, \quad (24)$$

where  $(\theta, J)$  are the angle-action coordinates and  $\Omega(J) \equiv d\mathcal{E}(J)/dJ$ , and where  $C^n(J)$  is defined by

$$C^n(J) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos q(\theta, J) e^{-in\theta} d\theta, \quad n \in \mathbb{Z}. \quad (25)$$

Before proving Proposition 2, we note that all distributions such as (23) are stationary solutions to the Vlasov equation.

The monotonicity of  $f_0$  in  $J$  is satisfied for stationary solutions which are obtained as solutions to a variational equation associated with an optimization problem such as the maximization of the entropy or the minimization of the free energy. Then, the assumption imposed on  $f_0$  in Proposition 2 is not too restrictive, and has some physical relevance. We note also that  $\tilde{f}_0'(J)/\Omega(J)$  is finite for all  $J$  since  $d\hat{f}_0(\mathcal{E})/d\mathcal{E}$  has no singularity.

The proof of Proposition 2 can be performed in a similar manner to that applied to Proposition 1, although the  $f_0$  is spatially inhomogeneous in the present proof. We divide the stability analysis into two, one of which deals with stability against the perturbation in the direction parallel to the order parameter  $\vec{M}_0 = (M_0, 0)^T$  and the other with stability against the perturbation in the direction perpendicular to  $\vec{M}_0$ .

We first analyze the stability against the perturbation in the direction parallel to the order parameter  $\vec{M}_0 = (M_0, 0)^T$ . The dispersion function in this case is put in the form

$$D_x(\omega) = 1 + 2\pi \sum_{m \in \mathbb{Z}} \int_L \frac{m \tilde{f}_0'(J)}{m\Omega(J) - \omega} |C^m(J)|^2 dJ, \quad \text{Im}\omega > 0 \quad (26)$$

and the dispersion relation is given by  $D_x(\omega) = 0$  [34]. When  $\text{Im}\omega > 0$ , the term of  $m = 0$  in (26) vanishes and Eq. (26) is arranged as

$$D_x(\omega) = 1 + 2\pi \sum_{m \in \mathbb{Z} \setminus \{0\}} \int_L \frac{\tilde{f}_0'(J)}{\Omega(J) - \omega/m} |C^m(J)|^2 dJ, \quad \text{Im}\omega > 0. \quad (27)$$

We here note that  $D_x(0)$  is defined as  $D_x(0) = \lim_{\epsilon \rightarrow 0+} D_x(i\epsilon)$ , and

$$D_x(0) = 1 + 2\pi \sum_{m \neq 0} \int_L \frac{\tilde{f}_0'(J)}{\Omega(J)} |C^m(J)|^2 dJ \quad (28)$$

since the integrand in it has no singularity.

If there exists  $\omega$  such that  $D_x(\omega) = 0$  with  $\text{Im}\omega > 0$ , then one has  $\text{Im}D_x(\omega) = 0$ , which is written out as

$$\begin{aligned} \text{Im}D_x(\omega) &= \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{2\pi}{m} \omega_i \int_L \frac{|C^m(J)|^2 \Omega(J) \tilde{f}_0'(J)}{[\Omega(J) - \omega_r/m]^2 + (\omega_i/m)^2} dJ \\ &= \sum_{m \in \mathbb{N}} \left[ \frac{2\pi}{m} \omega_i \int_L \frac{|C^m(J)|^2 \Omega(J) \tilde{f}_0'(J)}{[\Omega(J) - \omega_r/m]^2 + (\omega_i/m)^2} dJ - \frac{2\pi}{m} \omega_i \int_L \frac{|C^{-m}(J)|^2 \Omega(J) \tilde{f}_0'(J)}{[\Omega(J) + \omega_r/m]^2 + (\omega_i/m)^2} dJ \right] \\ &= \sum_{m \in \mathbb{N}} \frac{8\pi}{m} \omega_r \omega_i \int_L \frac{|C^m(J)|^2 \Omega(J) \tilde{f}_0'(J)}{\{[\Omega(J) - \omega_r/m]^2 + (\omega_i/m)^2\} \{[\Omega(J) + \omega_r/m]^2 + (\omega_i/m)^2\}} dJ = 0, \end{aligned} \quad (29)$$

where we have used the fact  $C^m(J) = C^{-m}(J)^*$  which is derived from (25). Since  $|C^m(J)|^2 > 0$  for some  $m \in \mathbb{N}$ , and  $\Omega(J) > 0$  and  $\tilde{f}_0'(J) < 0$  for all  $J$ , the integrals in (29) give negative values. Then, (29) yields  $\omega_r = 0$  since  $\omega_i > 0$ . We thus have shown that if  $\omega = \omega_r + i\omega_i$  with  $\omega_i > 0$  is a root of  $D_x(\omega) = 0$ , then  $\omega_r = 0$ .

On account of  $\omega_r = 0$ , the dispersion relation reduces to

$$D_x(i\omega_i) = 1 + 4\pi \sum_{m \in \mathbb{N}} \int_L \frac{\Omega(J) \tilde{f}_0'(J)}{\Omega(J)^2 + (\omega_i/m)^2} |C^m(J)|^2 dJ = 0. \quad (30)$$

The function  $D_x(i\omega_i)$  is a strictly increasing continuous function of  $\omega_i$ , and converges to 1,  $D_x(i\omega_i) \rightarrow 1$ , as  $\omega_i \rightarrow \infty$ . This implies that if  $D_x(0) < 0$ , there is a positive number  $\omega_i > 0$  such that  $D_x(i\omega_i) = 0$ . Put another way, if  $D_x(0) < 0$ , there is an  $\omega$  such that  $D_x(\omega) = 0$ ,  $\text{Im}\omega > 0$ . The converse is also shown by taking the contraposition of that there is no root  $\omega$  of  $D(\omega)$  with  $\text{Im}\omega > 0$  if  $D_x(0) \geq 0$ . We hence conclude that there is no unstable eigenvalue for the perturbation whose direction is parallel to the order parameter  $\vec{M}_0 = (M_0, 0)^T$ , if and only if  $D_x(0) \geq 0$ . If  $D_x(0) = 0$ , the operator  $\hat{\mathcal{L}}$  has an embedded eigenvalue 0, so that  $f_0$  is neutrally spectrally stable.

To derive the spectral stability criterion (24), we have only to prove the relation  $D_x(0) = I[f_0]$ , which can be done by performing the same procedure as that carried out in Appendix C of [36]. According to Appendix B of [34], the function  $\cos q$  is expressed as

$$\cos q(\theta, J(k)) = \begin{cases} 1 - 2k^2 \text{sn}^2\left(\frac{2K(k)}{\pi}\theta, k\right), & k < 1 \\ 1 - 2\text{sn}^2\left(\frac{K(1/k)}{\pi}\theta, \frac{1}{k}\right), & k > 1 \end{cases} \quad (31)$$

where  $K(k)$  is the complete elliptic integral of the first kind [40], and where  $k$  is defined as

$$k \equiv \sqrt{\frac{\mathcal{E}(J) + M_0}{2M_0}}. \quad (32)$$

Owing to the periodicity of the Jacobian elliptic functions [40], the function  $\cos q(\theta, J)$  is  $2\pi$  periodic with respect to  $\theta$ . Then, from Parseval's equality, we obtain

$$2\pi \sum_{m \neq 0} |C^m(J)|^2 = \int_{-\pi}^{\pi} \cos^2 q(\theta, J) d\theta - 2\pi |C^0(J)|^2. \quad (33)$$

By using this equation, we rewrite the second terms in the right-hand side of (28) as

$$\int_L dJ \frac{\tilde{f}'_0(J)}{\Omega(J)} \int_{-\pi}^{\pi} \cos^2 q(\theta, J) d\theta - 2\pi \int_L \frac{\tilde{f}'_0(J)}{\Omega(J)} |C^0(J)|^2 dJ. \quad (34)$$

Keeping in mind the fact that the stationary solution  $f_0$  to the Vlasov equation depends on the action  $J$  only through a single-body energy  $\mathcal{E}$ , we arrange the first term of (34) as

$$\begin{aligned} & \int_L dJ \frac{\tilde{f}'_0(J)}{\Omega(J)} \int_{-\pi}^{\pi} \cos^2 q(\theta, J) d\theta \\ &= \int_L dJ \int_{-\pi}^{\pi} \frac{d\hat{f}_0}{d\mathcal{E}}(\mathcal{E}(J)) \cos^2 q(\theta, J) d\theta \\ &= \iint_{\mu} \frac{d\hat{f}_0}{d\mathcal{E}}(\mathcal{E}(J)) \cos^2 q(\theta, J) d\theta dJ \\ &= \iint_{\mu} \frac{1}{p} \frac{\partial f_0}{\partial p}(q, p) \cos^2 q d q d p. \end{aligned} \quad (35)$$

In the course of analysis, we have used the fact that the transformation  $(\theta, J) \mapsto (q, p)$  is canonical. We note that  $d\hat{f}_0/d\mathcal{E}$  is assumed to be continuous in  $J$ , and hence the integration with respect to  $J$  is taken along on the real  $J$  axis. Then, by using (21) and (22), the second equality in (35) is

derived. Equations (28), (34), and (35) are put together to show the relation  $D_x(0) = I[f_0]$ .

So far, we have investigated the stability against perturbations in the direction parallel to the order parameter  $\vec{M}_0 = (M_0, 0)^T$ . We proceed to look into the stability against a perturbation in the direction perpendicular to the order parameter  $\vec{M}_0$ . The dispersion relation corresponding to the direction perpendicular to the order parameter  $\vec{M}_0$  is expressed as

$$D_y(\omega) = 1 + 2\pi \sum_{m \in \mathbb{Z}} \int_L \frac{m \tilde{f}'_0(J)}{m\Omega(J) - \omega} |S^m(J)|^2 dJ = 0, \quad (36)$$

for  $\text{Im}\omega > 0$ , where

$$S^m(J) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin q(\theta, J) e^{-im\theta} d\theta. \quad (37)$$

We note that  $S^0(J) = 0$ . In fact,  $\sin q(\theta, J)$  is expressed as

$$\sin q(\theta, J(k)) = \begin{cases} 2k \text{sn}\left(\frac{2K(k)}{\pi}\theta, k\right) \text{dn}\left(\frac{2K(k)}{\pi}\theta, k\right), & k < 1 \\ 2\text{sn}\left(\frac{K(1/k)}{\pi}\theta, \frac{1}{k}\right) \text{cn}\left(\frac{K(1/k)}{\pi}\theta, \frac{1}{k}\right), & k > 1 \end{cases} \quad (38)$$

it is odd with respect to  $\theta$  for all  $J$  [40], so that one has  $S^0(J) = 0$ .

Following the same procedure as that for proving the relation  $D_x(0) = I[f_0]$  and taking into account the relation  $S^0(J) = 0$ , we obtain

$$D_y(0) = 1 + \iint_{\mu} \frac{1}{p} \frac{\partial f_0}{\partial p}(q, p) \sin^2 q d q d p. \quad (39)$$

If  $D_y(0) \geq 0$ , there is no eigenmode which brings about the instability in a direction perpendicular to the order parameter  $\vec{M}_0$ . Actually, the equality  $D_y(0) = 0$  is satisfied for any stationary solution subject to the assumptions in Proposition 2 with (16), which is proved as follows [23]:

$$\begin{aligned} & \iint_{\mu} \frac{1}{p} \frac{\partial f_0}{\partial p}(q, p) \sin^2 q d q d p \\ &= \iint_{\mu} \frac{d\hat{f}_0}{d\mathcal{E}}(\mathcal{E}(q, p)) \sin^2 q d q d p \\ &= \frac{1}{M_0} \iint_{\mu} \frac{\partial f_0}{\partial q}(q, p) \sin q d q d p \\ &= -\frac{1}{M_0} \iint_{\mu} f_0(q, p) \cos q d q d p = -1. \end{aligned} \quad (40)$$

Since  $D_x(0) = I[f_0]$  and  $D_y(0) = 0$ , we have obtained the spectral stability criterion (24) for the spatially inhomogeneous solutions to the Vlasov equation.

It is to be remarked that any spectrally stable solution which is spatially inhomogeneous is neutrally spectrally stable since there is an embedded eigenvalue 0 which comes from  $D_y(0) = 0$ . To compute  $D_x(0)$  or the right-hand side of (24), we should express  $C^0(J)$  in terms of known functions. On using the explicit expression of  $\Omega(J)$  and  $C^0(J)$  given, respectively, in Appendix B of [34] and Appendix C of [36],  $D_x(0)$  is described

explicitly as

$$D_x(0) = 1 + \iint_{\mu} \frac{1}{p} \frac{\partial f_0}{\partial p}(q, p) \cos^2 q \, dq \, dp \\ - \frac{4}{\sqrt{M_0}} \int_0^1 K(k) \left[ \frac{2E(k)}{K(k)} - 1 \right]^2 \tilde{f}_0'(k) dk \\ - \frac{4}{\sqrt{M_0}} \int_1^{\infty} \frac{K(1/k)}{k} \left[ \frac{2k^2 E(1/k)}{K(1/k)} + 1 - 2k^2 \right]^2 \\ \times \tilde{f}_0'(k) dk, \quad (41)$$

where  $\tilde{f}_0(k) \equiv \tilde{f}_0(J(k))$ , and where  $E(k)$  is the complete elliptic integral of the second kind [40].

### C. Stationary states realized as critical points of some invariant functionals

We will give a necessary and sufficient condition of the formal stability of a stationary state. To look into the formal stability, we introduce invariant functionals. The Vlasov dynamics satisfies the normalization condition

$$\mathcal{N}[f] \equiv \iint_{\mu} f(q, p) \, dq \, dp = 1, \quad (42)$$

the momentum conservation law

$$\mathcal{P}[f] \equiv \iint_{\mu} p f(q, p) \, dq \, dp = 0, \quad (43)$$

and the energy conservation law

$$\mathcal{U}[f] \equiv \iint_{\mu} \frac{p^2}{2} f(q, p) \, dq \, dp - \frac{1}{2} (M_x[f]^2 + M_y[f]^2) = U, \quad (44)$$

where  $U$  is a fixed value. The Vlasov dynamics additionally has an infinite number of Casimir invariants denoted by

$$\mathcal{S}[f] = \iint_{\mu} s(f(q, p)) \, dq \, dp. \quad (45)$$

We here assume that  $s$  is a strictly concave and twice differentiable function defined for the non-negative real numbers.

We will look into the formal stability of the stationary solution realized as the critical point of (45) under constraints (42), (43), and (44). A critical point  $\tilde{f}_0(J)$  is a solution to the variational equation

$$\delta \mathcal{F} = \delta(\mathcal{S} - \beta \mathcal{U} - \alpha \mathcal{N}) = 0, \quad (46)$$

which is written out as

$$s'(\tilde{f}_0(J)) = \beta E(J) + \alpha, \quad (47)$$

where  $\alpha$  and  $\beta$  are Lagrangian multipliers. Since  $s(x)$  is a strictly concave differentiable function defined on  $x \geq 0$ , its derivative  $s'(x)$  is strictly decreasing on  $x \geq 0$ , and the inverse function  $(s')^{-1}(y)$  exists and is strictly decreasing on the range of the function  $s'$ . We are then allowed to put the solution  $\tilde{f}_0(J)$  to the variational equation (46) in the form

$$\hat{f}_0(\mathcal{E}) = \tilde{f}_0(J(\mathcal{E})) = (s')^{-1}(\beta \mathcal{E} + \alpha). \quad (48)$$

The parameter  $\beta$  is positive [23]. To see this, we assume that  $\beta$  were not positive. (i) When  $\beta < 0$ , from (48), the

function  $\hat{f}_0(\mathcal{E})$  is strictly increasing with respect to  $\mathcal{E}$ , so that the function  $\tilde{f}_0(J)$  is strictly increasing with respect to  $J$ . (ii) When  $\beta = 0$ ,  $\hat{f}_0(\mathcal{E})$  is a constant for the whole  $\mathcal{E}$ , so that  $\tilde{f}_0(J)$  is a constant for the whole  $J$ . In these cases, the integral  $\int_L \tilde{f}_0(J) dJ$  diverges, and hence  $\tilde{f}_0(J)$  can not be a probabilistic density function. Hence, parameter  $\beta$  must be positive. In the case  $\beta > 0$ ,  $\tilde{f}_0(J)$  can be a probabilistic density function.

Since  $\beta$  is shown to be positive, and since  $s$  is strictly concave, a solution (48) to the variational equation (46) is a stationary solution to the Vlasov equation satisfying  $d\hat{f}_0/d\mathcal{E} < 0$  and  $d\tilde{f}_0/dJ < 0$ .

### D. Formal stability criterion in the most refined form

In this section, we look into the most refined formal stability of the spatially inhomogeneous stationary solution  $f_0$  which is a critical point of the functional (45) under the constraint conditions (42), (43), and (44). To start with, we note that  $C^n(J) = C^{-n}(J)$ . In fact, from  $\text{sn}(u, k) = -\text{sn}(-u, k)$  [40] and (31), one has that  $\cos q(\theta, J)$  is even with respect to  $\theta$ , so that  $C^n(J)$  is shown to be real from the definition (25) and  $C^n(J) = C^{-n}(J)$ , and further  $|C^n(J)|^2 = C^n(J)^2$ .

We derive the formal stability criterion for spatially inhomogeneous solutions on the basis of the following claim.

*Claim 3.* A solution  $\tilde{f}_0(J)$  to the variational equation (46) is formally stable, if and only if the second-order variation of the functional  $\mathcal{F} = \mathcal{S} - \beta \mathcal{U} - \alpha \mathcal{N}$  is negative definite at  $\tilde{f}_0$  under the constraint of the Casimir invariants. That is,  $\delta^2 \mathcal{F}[\tilde{f}_0][\delta f, \delta f] < 0$  for any nonzero variation  $\delta f$  leaving invariant the functional of the form (8) up to first order for any function  $Q$ .

To investigate the condition  $\delta^2 \mathcal{F}[\tilde{f}_0][\delta f, \delta f] < 0$ , we start by putting the function  $\gamma$  as

$$\gamma(J) = \frac{\beta}{s''(\tilde{f}_0(J))} = \frac{\tilde{f}_0'(J)}{\Omega(J)} = \frac{d\hat{f}_0}{d\mathcal{E}}(\mathcal{E}(J)). \quad (49)$$

Then, the second-order variation of  $\mathcal{F}$  is described as

$$\delta^2 \mathcal{F}[\tilde{f}_0][\delta \tilde{f}, \delta \tilde{f}] = \iint_{\mu} \frac{\beta}{\gamma(J)} \delta \tilde{f}(\theta, J)^2 d\theta \, dJ \\ + \beta \left[ \iint_{\mu} \cos q(\theta, J) \delta \tilde{f}(\theta, J) d\theta \, dJ \right]^2 \\ + \beta \left[ \iint_{\mu} \sin q(\theta, J) \delta \tilde{f}(\theta, J) d\theta \, dJ \right]^2. \quad (50)$$

On account of the constraints of the Casimir invariants (8) up to first order, the perturbation should satisfy the constraint

$$Q[f_0 + \delta f] - Q[f_0] = \iint_{\mu} Q'(f_0(q, p)) \delta f(q, p) \, dq \, dp \\ = \int_L dJ Q'(\tilde{f}_0(J)) \int_{-\pi}^{\pi} \delta \tilde{f}(\theta, J) d\theta \\ = 0. \quad (51)$$



Since  $Q$  is chosen arbitrarily, we can look on  $Q'(\tilde{f}_0(J))$  as a function of  $J$  (or  $\mathcal{E}(J)$ ) chosen arbitrarily. We are then allowed to restrict perturbations to those satisfying

$$\int_{-\pi}^{\pi} \delta \tilde{f}(\theta, J) d\theta = 0, \quad \forall J. \quad (52)$$

We now divide the perturbation  $\delta \tilde{f}(\theta, J)$  into even and odd parts with respect to  $\theta$ ,

$$\delta \tilde{f}(\theta, J) = \delta_e \tilde{f}(\theta, J) + \delta_o \tilde{f}(\theta, J), \quad (53)$$

where

$$\begin{aligned} \delta_e \tilde{f}(\theta, J) &= \frac{1}{2} [\delta \tilde{f}(\theta, J) + \delta \tilde{f}(-\theta, J)], \\ \delta_o \tilde{f}(\theta, J) &= \frac{1}{2} [\delta \tilde{f}(\theta, J) - \delta \tilde{f}(-\theta, J)]. \end{aligned} \quad (54)$$

When  $\delta \tilde{f}$  in the functional (50) is replaced by (53), the functional (50) is arranged as

$$\begin{aligned} \delta^2 \mathcal{F}[\tilde{f}_0][\delta \tilde{f}, \delta \tilde{f}] &= \iint_{\mu} \frac{\beta}{\gamma(J)} \delta_e \tilde{f}(\theta, J)^2 d\theta dJ \\ &+ \beta \left[ \iint_{\mu} \cos q(\theta, J) \delta_e \tilde{f}(\theta, J) d\theta dJ \right]^2 \\ &+ \iint_{\mu} \frac{\beta}{\gamma(J)} \delta_o \tilde{f}(\theta, J)^2 d\theta dJ \\ &+ \beta \left[ \iint_{\mu} \sin q(\theta, J) \delta_o \tilde{f}(\theta, J) d\theta dJ \right]^2 \\ &= \delta^2 \mathcal{F}[\tilde{f}_0][\delta_e \tilde{f}, \delta_e \tilde{f}] + \delta^2 \mathcal{F}[\tilde{f}_0][\delta_o \tilde{f}, \delta_o \tilde{f}], \end{aligned} \quad (55)$$

where we have used the fact that

$$M_x[\delta_o \tilde{f}] = 0, \quad M_y[\delta_e \tilde{f}] = 0, \quad (56)$$

which come from the fact that  $\cos q(\theta, J)$  (resp.  $\sin q(\theta, J)$ ) is even (resp. odd) with respect to  $\theta$  on account of (31) (resp. (38)). Equation (55) means that  $\delta_e \tilde{f}$  and  $\delta_o \tilde{f}$  are not coupled in (55). As for the second term in the right-hand side of the last equality in (55), we recall that spatially inhomogeneous stationary solutions are already known to be neutrally formally stable against a perturbation  $\delta_o \tilde{f}$  whose direction is perpendicular to the direction of the order parameter  $\vec{M}_0$ , as is shown in [23]. This fact is consistent with the fact that the order parameter may rotate if an arbitrarily small external field is turned on perpendicularly to the order parameter [36]. We do not take into account this rotation as long as we treat a formal stability of the stationary solution  $f_0$ , as we mentioned in Sec. III. On account of (56), we are now left with the analysis of  $\delta^2 \mathcal{F}[\tilde{f}_0][\delta_e \tilde{f}, \delta_e \tilde{f}]$ , the integrals in (55) for the even part  $\delta_e \tilde{f}$  whose direction is parallel to the order parameter  $\vec{M}_0$ .

In what follows, we prove the following proposition:

*Proposition 4.* Let  $f_0$  be a solution to the variational equation (46). The inequality

$$I[f_0] = D_x(0) > 0 \quad (57)$$

is equivalent to the condition

$$\delta^2 \mathcal{F}[\tilde{f}_0][\delta_e \tilde{f}, \delta_e \tilde{f}] < 0 \quad (58)$$

for any  $\delta_e \tilde{f} \neq 0$  under the constraint (52). Therefore, the inequality (57) is a necessary and sufficient condition for the formal stability of  $\tilde{f}_0$ .

In the situation stated so far, the second-order variation (50) is put in the form

$$\begin{aligned} \delta^2 \mathcal{F}[\tilde{f}_0][\delta_e \tilde{f}, \delta_e \tilde{f}] &= \iint_{\mu} \frac{\beta}{\gamma(J)} \delta_e \tilde{f}(\theta, J)^2 d\theta dJ \\ &+ \beta \left[ \iint_{\mu} \cos q(\theta, J) \delta_e \tilde{f}(\theta, J) d\theta dJ \right]^2. \end{aligned} \quad (59)$$

We first show that a nonzero  $\delta_e \tilde{f}$  satisfying  $M_x[\delta_e \tilde{f}] = 0$  does not bring about the formal instability. Indeed, (59) becomes

$$\delta^2 \mathcal{F}[\tilde{f}_0][\delta_e \tilde{f}, \delta_e \tilde{f}] = \iint_{\mu} \frac{\beta}{\gamma(J)} \delta_e \tilde{f}(\theta, J)^2 d\theta dJ, \quad (60)$$

and is negative since  $\gamma(J) < 0$  and  $\beta > 0$ , as was mentioned in Sec. VC.

We proceed to perform the stability analysis with the constraint condition

$$M_x[\delta_e \tilde{f}] = \iint_{\mu} \cos q(\theta, J) \delta_e \tilde{f}(\theta, J) d\theta dJ = 1. \quad (61)$$

We note that the value of  $M_x[\delta_e \tilde{f}]$  can be chosen arbitrary because this value changes only the scaling of (59) and does not change the sign of (59). We expand the perturbation  $\delta_e \tilde{f}$  into the Fourier series in  $\theta$ :

$$\delta_e \tilde{f}(\theta, J) = \sum_{n \neq 0} \hat{f}_n^e(J) e^{in\theta}, \quad \hat{f}_n^e(J) = \hat{f}_{-n}^e(J). \quad (62)$$

We note that the 0th Fourier mode vanishes thanks to the constraint condition (52). Substituting (62) into (59), we obtain the functional in  $\{\hat{f}_n^e\}_{n \neq 0}$ :

$$\begin{aligned} \mathcal{G}_e[\{\hat{f}_n^e\}_{n \neq 0}] &\equiv \frac{1}{2\pi} \delta^2 \mathcal{F}[\tilde{f}_0][\delta_e \tilde{f}, \delta_e \tilde{f}] = \sum_{n \neq 0} \int_L \frac{\beta}{\gamma(J)} \hat{f}_n^e(J)^2 \\ &\times dJ + 2\pi\beta \left( \sum_{m \neq 0} \int_L C^m(J') \hat{f}_m^e(J') dJ' \right)^2. \end{aligned} \quad (63)$$

We look for a critical point of  $\mathcal{G}_e$  under the constraint (61) which is rewritten in terms of  $\{\hat{f}_n^e\}_{n \neq 0}$  as

$$\mathcal{M}_x[\{\hat{f}_m^e\}_{m \neq 0}] \equiv 2\pi \sum_{m \neq 0} \int_L C^m(J) \hat{f}_m^e(J) dJ = 1. \quad (64)$$

The functional  $\mathcal{G}_e[\{\hat{f}_m^e\}_{m \neq 0}]$  takes a critical value under the constraint condition (64) if

$$\delta_n \mathcal{G}_e[\{\hat{f}_m^e\}_{m \neq 0}] - \eta \delta_n \mathcal{M}_x[\{\hat{f}_m^e\}_{m \neq 0}] = 0, \quad n \in \mathbb{Z} \setminus \{0\} \quad (65)$$

where  $\eta$  is a Lagrangian multiplier, and  $\delta_n G_e$  is defined by

$$\begin{aligned} \delta_n G_e[\{\hat{f}_m^e\}_{m \neq 0}] &\equiv G_e[\{\hat{f}_m^e + \delta \hat{f}_n^e \delta_{mn}\}_{m \neq 0}] - G_e[\{\hat{f}_m^e\}_{m \neq 0}] \\ &= 2\beta \int_L \delta \hat{f}_n^e(J) \left[ \frac{\hat{f}_n^e(J)}{\gamma(J)} + 2\pi C^n(J) \right. \\ &\quad \left. \times \left( \sum_{m \neq 0} \int_L C^m(J') \hat{f}_m^e(J') dJ' \right) \right] dJ, \quad (66) \end{aligned}$$

where  $\delta_{mn}$  is the Kronecker delta. Hence, Eq. (65) results in

$$\begin{aligned} \hat{f}_n^e(J) &= - \left( 2\pi \sum_{m \neq 0} \int_L C^m(J') \hat{f}_m^e(J') dJ' \right) C^n(J) \gamma(J) \\ &\quad + \frac{\pi \eta}{\beta} C^n(J) \gamma(J) = -\xi C^n(J) \gamma(J) \quad (67) \end{aligned}$$

for all  $n \in \mathbb{Z} \setminus \{0\}$ , where we have used (64) and put  $\xi \equiv 1 - \pi \eta / \beta$ . Substituting (67) into (64), we obtain the value of  $\xi$  as

$$\xi = \frac{-1}{2\pi \sum_{m \neq 0} \int_L C^m(J)^2 \gamma(J) dJ}. \quad (68)$$

A nonvanishing critical point  $\{\hat{f}_n^{e,m}\}_{n \in \mathbb{Z} \setminus \{0\}}$  is therefore given by

$$\hat{f}_n^{e,m}(J) = \frac{C^n(J) \gamma(J)}{2\pi \sum_{m \neq 0} \int_L C^m(J')^2 \gamma(J') dJ'}, \quad n \in \mathbb{Z} \setminus \{0\}. \quad (69)$$

Substituting (69) into (63), we obtain

$$\begin{aligned} G_e[\{\hat{f}_n^{e,m}\}_{n \neq 0}] &= \frac{\beta}{4\pi^2 \sum_{m \neq 0} \int_L C^m(J)^2 \gamma(J) dJ} + \frac{\beta}{2\pi} \\ &= \frac{\beta}{4\pi^2 \sum_{m \neq 0} \int_L C^m(J)^2 \gamma(J) dJ} \\ &\quad \times \left[ 1 + 2\pi \sum_{l \neq 0} \int_L \frac{\tilde{f}_0^l(J)}{\Omega(J)} C^l(J)^2 dJ \right], \quad (70) \end{aligned}$$

where we have used (49).

Since  $\gamma(J) < 0$ , and since  $C^n(J) \neq 0$  for some  $J$  and  $n \in \mathbb{Z} \setminus \{0\}$ , we have

$$\sum_{m \neq 0} \int_L C^m(J)^2 \gamma(J) dJ < 0. \quad (71)$$

It then follows, from (70) along with (71) and the positivity of  $\beta$  which has been shown at the end of Sec. VB, that the quadratic form (63) is negative definite if and only if the inequality

$$D_x(0) = 1 + 2\pi \sum_{l \neq 0} \int_L \frac{\tilde{f}_0^l(J)}{\Omega(J)} C^l(J)^2 dJ > 0 \quad (72)$$

is satisfied. We hence conclude that the inequality (57) is a necessary and sufficient condition for formal stability. Once the criterion (57) is obtained, we no longer have to seek an infinite number of Lagrangian multipliers to get the most refined formal stability criterion given in [23]. The formal stability criterion (57) is stronger than the condition that  $\tilde{f}_0(J)$

is spectrally stable in the sense that the equality in (24) is not allowed.

*Remark.* We have shown that the stability of  $f_0$  is determined by the sign of  $I[f_0] = D_x(0)$ . Further, the value of the positive  $I[f_0]$  is thought to express a strength of stability of  $f_0$  since the zero-field isolated-susceptibility  $\chi$  is derived as

$$\chi = \frac{1 - D_x(0)}{D_x(0)} = \frac{1 - I[f_0]}{I[f_0]}, \quad (73)$$

with the linear response theory based on the Vlasov equation [36]. Equation (73) implies that stability of a stationary state  $f_0$  becomes stronger as  $I[f_0]$  becomes larger since  $I[f_0] = D_x(0) \leq 1$ . The last inequality is derived as follows. As we mentioned in Sec. VB, the function  $D_x(i\omega_i)$  is strictly increasing and continuous with respect to  $\omega_i \geq 0$ , and  $D_x(i\omega_i) \rightarrow 1$ , as  $\omega_i \rightarrow \infty$ .

### E. Observation of the criteria

Let us observe what kinds of stationary states are likely to be stable through the stability analysis for a family of the stationary water-bag distributions [34]

$$f_{wb}(q, p) = \eta_0 \Theta(\mathcal{E}^* - \mathcal{E}(q, p)), \quad \mathcal{E}(q, p) = \frac{p^2}{2} - M_0 \cos q, \quad (74)$$

where  $\Theta$  is the Heaviside step function. Although the water-bag distributions (74) do not satisfy assumptions in Proposition 2, they make it possible to observe the stability visually. Let us put  $k^* = \sqrt{(\mathcal{E}^* + M_0)/(2M_0)}$ . For each fixed  $M_0$ , the two parameters  $\eta_0$  and  $\mathcal{E}^*$  are determined by the normalization condition

$$\begin{aligned} 1 &= \iint_{\mu} f_{wb}(q, p) dq dp \\ &= \begin{cases} 16\eta_0 \sqrt{M_0} [E(k^*) - (1 - k^{*2})K(k^*)], & k^* < 1 \\ 16\eta_0 \sqrt{M_0} k^* E(1/k^*), & k^* > 1 \end{cases} \quad (75) \end{aligned}$$

and the self-consistent equation

$$\begin{aligned} M_0 &= \iint_{\mu} \cos q f_{wb}(q, p) dq dp \\ &= \begin{cases} 1 - \frac{2}{3} \frac{(2 - k^{*2})E(k^*) - (2 - 2k^{*2})K(k^*)}{E(k^*) - (1 - k^{*2})K(k^*)}, & k^* < 1 \\ \frac{2k^{*2} - 1}{3} - \frac{2k^{*2} - 2}{3} \frac{K(1/k^*)}{E(1/k^*)}, & k^* > 1. \end{cases} \quad (76) \end{aligned}$$

For the water-bag distribution (74), we are able to compute  $I[f_{wb}]$  explicitly by using equations

$$\frac{1}{p} \frac{\partial f_{wb}}{\partial p}(q, p) = \frac{d \hat{f}_{wb}}{d \mathcal{E}}(\mathcal{E}(q, p)) = -\eta_0 \delta(\mathcal{E}^* - \mathcal{E}(q, p)) \quad (77)$$

and

$$\frac{d \tilde{f}_0}{dk}(k) = 4M_0 k \frac{d \hat{f}_0}{d \mathcal{E}}(\mathcal{E}(k)) = -4M_0 k \eta_0 \delta(\mathcal{E}^* - \mathcal{E}(k)), \quad (78)$$

and by using Eq. (41). Then,  $I[f_{wb}]$  is written as

$$I[f_{wb}] = \begin{cases} 1 + \frac{(8k^{*2}-4)E(k^*)+(1-4k^{*2})K(k^*)}{12M_0[E(k^*)-(1-k^{*2})K(k^*)]} + \frac{[2E(k^*)-K(k^*)]^2}{4M_0K(k^*)[E(k^*)-(1-k^{*2})K(k^*)]}, & k^* < 1 \\ 1 + \frac{(8k^{*4}-4k^{*2})E(1/k^*)-(8k^{*4}-8k^{*2}+3)K(1/k^*)}{12M_0k^{*2}E(1/k^*)} + \frac{[2k^{*2}E(1/k^*)+(1-2k^{*2})K(1/k^*)]^2}{4M_0k^{*2}K(1/k^*)E(1/k^*)}, & k^* > 1. \end{cases} \quad (79)$$

Since  $\mathcal{E}^*$  and  $k^*$  is determined by  $M_0$ , then  $I[f_{wb}]$  in (79) can be looked on as a function of  $M_0$  and it is plotted in Fig. 2. According to this graph, the water bag  $f_{wb}$  is formally (resp. spectrally) stable when  $M_0 > M_0^c \simeq 0.369\,942$  (resp. when  $M_0 \geq M_0^c$ ). The critical value  $M_0^c \simeq 0.369\,942$  is obtained by solving the self-consistent equation (76) and  $I[f_{wb}] = 0$  simultaneously, and it is close to the estimation  $M_0^c \simeq 0.37$  in [34]. The water-bag distribution with large  $M_0$  tends to be stable, and the stability of it tends to be strong since  $I[f_{wb}]$  is monotonically increasing with respect to  $M_0$ , when  $M_0 > M_0^s \simeq 0.33$ . We illustrate it in Fig. 3. The water-bag distributions illustrated in Figs. 3(a) or 3(b) are unstable, and the one illustrated in Fig. 3(c) is stable.

## VI. COMPARISON WITH THE CANONICAL FORMAL STABILITY

Let us compare the formal stability criterion (57),  $I[f_0] = D_x(0) > 0$ , with the canonical formal stability criterion given in [23]. We start with a brief review of the canonical formal stability.

### A. Canonical formal stability

*Claim* ([23]). A solution  $f_0$  to the variational equation (46) is called canonically formally stable against any perturbation  $\delta_e f$  whose direction is parallel to the order parameter  $\vec{M}_0 = (M_0, 0)^T$ , if and only if the second-order variation of the functional  $\mathcal{F} = \mathcal{S} - \beta\mathcal{E} - \alpha\mathcal{N}$  at  $f_0$ ,  $\delta^2 \mathcal{F}[f_0]$ , subject to the normalization condition is negative definite, i.e.,

$$\delta^2 \mathcal{F}[f_0][\delta_e f, \delta_e f] < 0, \quad (80)$$

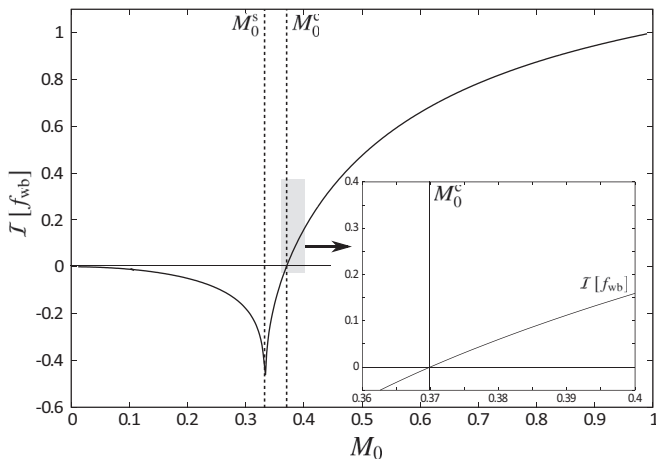


FIG. 2. Plot of  $I[f_{wb}]$  as a function of  $M_0$ :  $I[f_{wb}] > 0$  for  $M_0 > M_0^c$ . The critical value  $M_0^c \simeq 0.369\,942$ . The edge of the water bag  $\mathcal{E}(q, p) = \mathcal{E}^*$  coincides with the separatrix when  $M_0 = M_0^s \simeq 0.33$ .

for all  $\delta_e f \neq 0$  satisfying

$$\iint_{\mu} \delta_e f(q, p) dq dp = 0. \quad (81)$$

In particular, for the HMF model, the spatially inhomogeneous solution  $f_0(q, p)$  is canonically formally stable if and only if

$$I_C[f_0] \equiv 1 + \int_{-\pi}^{\pi} dq \cos^2 q \int_{-\infty}^{\infty} \frac{1}{p} \frac{\partial f_0}{\partial p}(q, p) dp - \frac{(\int_{-\pi}^{\pi} dq \cos q \int_{-\infty}^{\infty} \frac{1}{p} \frac{\partial f_0}{\partial p}(q, p) dp)^2}{\int_{-\pi}^{\pi} dq \int_{-\infty}^{\infty} \frac{1}{p} \frac{\partial f_0}{\partial p}(q, p) dp} > 0. \quad (82)$$

Satisfying the inequality (82) is sufficient but not necessary for the formal stability. We will show the existence of stationary solutions  $f_0$  which are not canonically formally stable, but formally stable in the most refined sense.

### B. Example: Family of distributions having metastable states

In this section, we prove the following proposition:

*Proposition 6.* Let  $D$  be a subset of  $\mathbb{R}^n$ . Assume that a family of smooth stationary solutions  $\mathcal{X} = \{f_0(q, p; M_0, \lambda) | \lambda \in D\}$ , which are parametrized with the order parameter  $M_0 = M_x[f_0]$  and a set of macroscopic quantities  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in D$ , such that there exists  $f_0^b(q, p) = f_0(q, p; M_0^b, \lambda^b) \in \mathcal{X}$  satisfying  $I[f_0^b] = 0$  and  $M[f_0^b] > 0$ . Moreover, assume that both  $I[f_0](M_0, \lambda)$  and  $I_C[f_0](M_0, \lambda)$  depend on  $M_0$  and  $\lambda$  continuously. Then, there are stationary solutions  $f_0 \in \mathcal{X}$  which do not satisfy the canonical formal stability criterion  $I_C[f_0](M_0, \lambda) > 0$  [Eq. (82)], but do satisfy the formal stability criterion  $I[f_0](M_0, \lambda) = D_x(0) > 0$  [Eq. (57)].

*Remark.* If the system has a first-order phase transition and a two-phase coexistence region in a parameter space  $(M_0, \lambda)$ , then we can take a family  $\mathcal{X}$  of stationary solutions

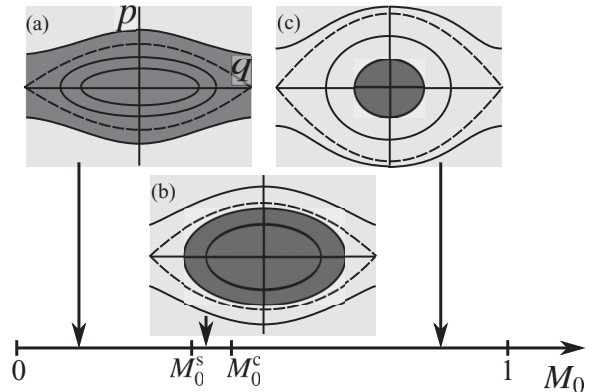


FIG. 3. Gray rectangles are  $\mu$  spaces for each  $M_0$ . The curves in  $\mu$  spaces are iso- $\mathcal{E}$  lines, and the broken curves are separatrices. On the dark gray region in each  $\mu$  space, the water-bag distribution takes nonzero value  $\eta_0$ .

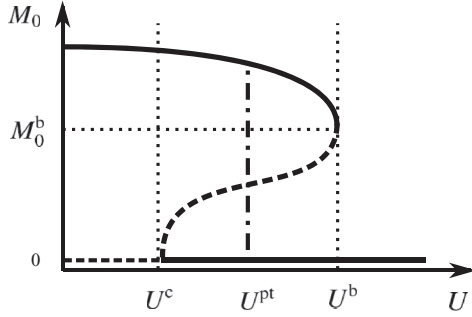


FIG. 4. Schematic picture of the phase diagram on  $(M_0, U)$  for some fixed  $M_I$  [28]. The solid curve represents the stable or metastable states which are realized as the local maximum points of the entropy. The broken curve represents the unstable states which are realized as the local minimum points of the entropy. These two curves meet at  $(M_0^b, U^b)$ . A region between  $U^c$  and  $U^b$  is called the two-phase coexistence region, and the first-order phase transition occurs at  $U^{pt}$ .

satisfying assumptions in Proposition 6. An example of such a family  $\mathcal{X}$  is known in Lynden-Bell's distributions (or Fermi-Dirac-type distributions) [41]. Within the Lynden-Bell's statistical mechanics with two-valued water-bag initial conditions, single-body distributions are parametrized with the order parameter in stationary states  $M_0$ , the energy  $U$ , and the parameter  $M_I$  describing to what extent particles spread on the  $\mu$  space before violent relaxation occurs. In this case, one has  $n = 2$  and  $(\lambda_1, \lambda_2) = (U, M_I)$  [28–30]. A schematic picture of the phase diagram  $(M_0, U, M_I)$  is exhibited in Fig. 4. On the three-dimensional parameter space, one can observe a first-order phase transition, a tricritical point, and a two-phase coexistence region.

We will omit the parameters  $(M_0, \lambda)$  from the description of  $f_0$ ,  $I[f_0]$ , and  $I_C[f_0]$  as long as no confusion arises. To prove the proposition, we first rewrite the third term of the right-hand side of (82) in terms of the angle-action coordinates

$$\frac{\left( \int_{-\pi}^{\pi} dq \cos q \int_{-\infty}^{\infty} \frac{1}{p} \frac{\partial f_0}{\partial p}(q, p) dp \right)^2}{\int_{-\pi}^{\pi} dq \int_{-\infty}^{\infty} \frac{1}{p} \frac{\partial f_0}{\partial p}(q, p) dp} = 2\pi \frac{\left( \int_L \frac{\tilde{f}_0'(J)}{\Omega(J)} C^0(J) dJ \right)^2}{\int_L \frac{\tilde{f}_0(J)}{\Omega(J)} dJ}. \quad (83)$$

The difference between  $I[f_0]$  and  $I_C[f_0]$  is calculated as

$$\begin{aligned} I[f_0] - I_C[f_0] &= -2\pi \int_L \frac{\tilde{f}_0'(J)}{\Omega(J)} C^0(J)^2 dJ \\ &\quad + 2\pi \frac{\left( \int_L \frac{\tilde{f}_0'(J)}{\Omega(J)} C^0(J) dJ \right)^2}{\int_L \frac{\tilde{f}_0(J)}{\Omega(J)} dJ} \\ &= -2\pi \int_L \frac{\tilde{f}_0'(J)}{\Omega(J)} dJ' \left[ \int_L C^0(J)^2 P(J) dJ \right. \\ &\quad \left. - \left( \int_L C^0(J) P(J) dJ \right)^2 \right], \end{aligned} \quad (84)$$

where  $P(J)$  is defined to be

$$P(J) \equiv \frac{1}{\int_L \frac{\tilde{f}_0(J')}{\Omega(J')} dJ'} \frac{\tilde{f}_0'(J)}{\Omega(J)} \geq 0. \quad (85)$$

We note that the inequality

$$\int_L C^0(J)^2 P(J) dJ - \left( \int_L C^0(J) P(J) dJ \right)^2 \geq 0 \quad (86)$$

is satisfied for any  $f_0$ . In fact, on account of

$$\int_L P(J) dJ = 1, \quad (87)$$

we obtain the equation

$$\begin{aligned} \int_L C^0(J)^2 P(J) dJ - \left( \int_L C^0(J) P(J) dJ \right)^2 \\ = \int_L \left[ C^0(J) - \int_L C^0(J') P(J') dJ' \right]^2 P(J) dJ, \end{aligned} \quad (88)$$

which implies Eq. (86). If the equality holds in (86), Eq. (88) results in

$$C^0(J) = \int_L C^0(J') P(J') dJ' = \text{const}, \quad \forall J. \quad (89)$$

However, this equality can not be realized for any smooth spatially inhomogeneous solution, so that Eq. (86) should be

$$\int_L C^0(J)^2 P(J) dJ - \left( \int_L C^0(J) P(J) dJ \right)^2 > 0. \quad (90)$$

Equation (84) with  $\gamma(J) < 0$  and this inequality are put together to provide

$$I[f_0] > I_C[f_0] \quad (91)$$

for any smooth spatially inhomogeneous stationary solution  $f_0$ . This implies the known inclusion relation [23]

$$\begin{aligned} \{ \text{canonically formally stable states} \} \\ \cap \\ \{ \text{formally stable states} \}. \end{aligned} \quad (92)$$

We show that there is a solution which is formally stable but not canonically formally stable. From the assumption in Proposition 6,

$$I[f_0^b] = 0. \quad (93)$$

If one could decide the formal stability of a stationary solution correctly by using the canonical formal stability criterion (82) near the stationary solution  $f_0^b$ , the equation

$$I_C[f_0^b] = 0 \quad (94)$$

would be satisfied as well since  $I_C[f_0]$  depends on the parameters continuously. However, we have proved the inequality (91), so that (93) and (94) do not hold simultaneously. Then, the inequality

$$I_C[f_0^b] < I[f_0^b] = 0 \quad (95)$$

should be satisfied. From (91) and (95), it follows that there exists  $f_0$  such that

$$I_C[f_0] \leq 0, \quad I[f_0] > 0. \quad (96)$$

This implies that there is a solution which is formally stable, but not canonically formally stable.



## VII. SUMMARY AND DISCUSSION

We have worked out the spectral and formal stability criteria for spatially inhomogeneous stationary solutions to the Vlasov equation for the HMF model. These criteria are stated in the form of necessary and sufficient conditions (see Propositions 2 and 4). We stress that the assumptions for deriving the spectral stability criterion are satisfied by solutions to the variational equation (46). Our criterion avoids the problem of finding an infinite number of Lagrangian multipliers which are required in the previously obtained criterion [23]. We note that the formal stability criterion in Proposition 4 is stated in the form modified from the original one in [16] since the perturbation  $\delta_0 \tilde{f}$  perpendicular to the order parameter  $\tilde{M}_0$  with  $M_y[\delta_0 \tilde{f}] \neq 0$  brings about the neutral formal stability, and since the set of neutrally formally stable solutions is defined so as not to be included in the set of formally stable ones by [16].

We have interpreted the value of  $I[f_0] = D_x(0)$  as the strength of stability of the stable solutions. Further, we have observed that the stationary state with high density almost harmonic orbits tends to be stable, and its stability gets to be stronger as  $M_0$  gets large.

We have shown that stability of some solutions in the family of stationary solutions having two-phase coexistence region in the phase diagram can not be judged correctly by using the canonical formal stability criterion (see Proposition 6). A family of the Lynden-Bell's distributions is a family to which Proposition 6 is applied.

So far, we have analyzed stability criteria for the HMF model without external fields. The present methods can be applied for the HMF model with nonzero external field, if the Hamiltonian takes the form

$$H_N = \sum_{i=1}^N \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^N [1 - \cos(q_i - q_j)] - h \sum_{i=1}^N \cos q_i. \quad (97)$$

All we have to do is to modify the single-body energy (16) by adding to the potential  $-M_0 \cos q$  the term  $-h \cos q$  coming from external field. Then, we can make a similar discussion by using the angle-action coordinates. In this case, the rotational symmetry is broken, so that  $D_y(0) \neq 0$ . Hence, the spectral and the formal stability criteria become

$$D_x(0) \geq 0, \quad D_y(0) \geq 0, \quad (98)$$

and

$$D_x(0) > 0, \quad D_y(0) > 0, \quad (99)$$

respectively, and further the value of  $D_y(0)$  is computed as  $D_y(0) = h/(M_0 + h)$  by using the same procedure as in (40). In this case, the definition of formal stability is the same as that defined in [16], so that we can refer to the linear stability condition. Equation (98) is a necessary condition for the linear stability of the spatially inhomogeneous solution, and (99) is a sufficient condition of it. In fact, linearly stable states are spectrally stable states, and formally stable states are linearly stable states (see [16] for the proof). This discussion breaks down for the spatially inhomogeneous states in the HMF model without external field.

The stability analysis performed in this paper is applicable to the  $\alpha$ -HMF model ( $0 \leq \alpha < 1$ ) [42] with the Hamiltonian

$$H_N^\alpha = \sum_{i=1}^N \frac{p_i^2}{2} + \kappa_N^\alpha \sum_{i>j}^N \frac{1 - \cos(q_i - q_j)}{|r_i - r_j|^\alpha} - h \sum_{i=1}^N \cos q_i, \quad (100)$$

where  $r_i$  denotes the  $i$ th lattice point, and the lattice spacing is set as  $r_{i+1} - r_i = 1/N$ . We assume the periodic boundary condition for the lattice, and the distance  $|r_i - r_j|$  is actually  $\min\{|r_i - r_j|, 1 - |r_i - r_j|\}$ . Then,  $\kappa_N^\alpha$  is determined by

$$\sum_{i=1, i \neq j}^N \frac{\kappa_N^\alpha}{|r_i - r_j|^\alpha} = 1 \quad (101)$$

so that the system has the extensivity. Bachelard *et al.* [43] have derived the Vlasov equation describing the dynamics of the  $\alpha$ -HMF model in the limit of infinite  $N$ . If the stationary state  $f_0(q, p, r)$  does not depend on a configuration  $r$  on the lattice, then the dispersion function can be written explicitly, and we can derive the spectral and formal stability criteria for the  $\alpha$ -HMF model as for the HMF model.

Our procedure to look into the formal stability of the HMF model may be formally generalized to other models by using the biorthogonal functions and the Kalnajs' matrix form, which have been used in the astrophysics [5,35,44]. However, there are difficulties in extending our result for the HMF model to that for general models. For instance, finding an appropriate biorthogonal system and analyzing the Kalnajs' matrix form are hard tasks. In fact, the dispersion function is not a complex-valued function but a linear operator or a matrix. Hence, the formal stability criterion should be described in the form of positive definiteness of matrices or linear operators. If this matrix is a diagonal matrix or a block diagonal matrix with small blocks, we may get the formal stability criterion as for the HMF model for each diagonal element or each block.

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## APPENDIX A: DEFINITION OF THE NEUTRAL STABILITY

The neutral spectral stability is defined in terms of eigenvalues with vanishing real parts in Sec. II. However, the neutral spectral stability is originally defined in terms of spectra, not eigenvalues only, with vanishing real parts [16]. The reason why we modify the definition of neutral stability is that the linear operator  $\hat{\mathcal{L}}$  in (9) has always continuous spectrum on the imaginary axis, and this does not bring about spectral instability.

## APPENDIX B: EIGENVALUES OF THE LINEARIZED VLASOV OPERATOR AND ROOTS OF THE DISPERSION RELATION

We review the relation between eigenvalues of the linearized Vlasov operator (9) and roots of the dispersion relation after [39,45,46]. We do not deal with continuous spectra, embedded eigenvalues, or Landau poles in this appendix since they do not set off the spectral instability.

### 1. Spatially homogeneous state case

Let  $f_0(p)$  be a spatially homogeneous, even, unimodal, and smooth function. Let  $\hat{L}$  be the associated linearized Vlasov operator defined by (9). Then, the linearized Vlasov equation around  $f_0(p)$  takes the form

$$\frac{\partial f_1}{\partial t} = \hat{L} f_1. \quad (\text{B1})$$

We expand the both sides of (B1) into the Fourier series to find that the amplitude of the  $k$ th Fourier mode  $\hat{f}_{1,k}(p, t)$  obeys one of the following equations:

$$\frac{\partial \hat{f}_{1,k}}{\partial t} = \begin{cases} 0, & k = 0 \\ -ik \left[ p \hat{f}_{1,k}(p, t) + \pi \int_{-\infty}^{\infty} \hat{f}_{1,k}(p', t) dp' f'_0(p) \right], & |k| = 1 \\ -ik p \hat{f}_{1,k}(p, t), & |k| \geq 2. \end{cases} \quad (\text{B2})$$

For  $|k| \geq 2$ , there is no growth or damping mode. For  $|k| = 1$ , if  $\lambda$  is an eigenvalue of the linearized Vlasov operator  $\hat{L}$ , the associated eigenfunction can be written as  $\hat{f}_{1,k}(p) e^{\lambda t}$ , and we get the equation for  $\hat{f}_{1,k}(p)$ :

$$\hat{f}_{1,k}(p) = -\frac{\pi f'_0(p)}{p - i\lambda/k} \int_{-\infty}^{\infty} \hat{f}_{1,k}(p') dp'. \quad (\text{B3})$$

Integrating this equation over the whole  $\mathbb{R}$  results in

$$\int_{-\infty}^{\infty} \hat{f}_{1,k}(p') dp' \left[ 1 + \pi \int_{-\infty}^{\infty} \frac{f'_0(p)}{p - i\lambda/k} dp \right] = 0, \quad (\text{B4})$$

which is rewritten as

$$\Lambda(i\lambda/k) \int_{-\infty}^{\infty} \hat{f}_{1,k}(p') dp' = 0, \quad (\text{B5})$$

where  $\Lambda$  is defined to be

$$\Lambda(\omega) = 1 + \pi \int_{-\infty}^{\infty} \frac{f'_0(p)}{p - \omega} dp, \quad (\text{B6})$$

and is called the spectral function defined on  $\mathbb{C} \setminus \mathbb{R}$  [39]. In view of (12), we find that  $\Lambda(\omega) = D(\omega)$  on the upper half  $\omega$  plane. Here, we note that the relation between  $\Lambda(\omega)$  and  $D(\omega)$  for  $\omega \in \mathbb{C} \setminus \mathbb{R}$  [39] is given by

$$D(\omega) = \begin{cases} \Lambda(\omega), & \text{Im} \omega > 0 \\ \Lambda(\omega) + 2i\pi f'_0(\omega), & \text{Im} \omega < 0. \end{cases} \quad (\text{B7})$$

If the factor including  $\hat{f}_{1,k}$  in (B4) vanishes, i.e., if

$$\int_{-\infty}^{\infty} \hat{f}_{1,k}(p') dp' = 0, \quad (\text{B8})$$

then  $\hat{f}_{1,k}(p)$  vanishes owing to (B3), and thereby it has no concern with stability. We are then allowed to assume that

the left-hand side of (B8) does not vanish. It then follows from (B5) that if  $\lambda$  is an eigenvalue of the linearized Vlasov operator  $\hat{L}$ , the equation  $\Lambda(i\lambda/k) = 0$  should be satisfied for  $k = 1$  or  $-1$ . It is to be remarked that the assumptions imposed on  $f_0$  in Proposition 1 give rise to the relation

$$\Lambda(\omega) = \Lambda(-\omega) = \Lambda(\omega^*)^* = \Lambda(-\omega^*)^*. \quad (\text{B9})$$

This implies that if  $\omega$  with  $\text{Im} \omega > 0$  is a root of the dispersion relation (12), then the linearized Vlasov operator  $\hat{L}$  has eigenvalues  $i\omega$ ,  $-i\omega$ ,  $-i\omega^*$ , and  $i\omega^*$ . Therefore, if  $\hat{L}$  has an eigenvalue,  $\hat{L}$  has inevitably an unstable eigenvalue, so that the solution  $f_0$  should be unstable.

### 2. Spatially inhomogeneous state case

So far, we have analyzed the stability of spatially homogeneous states. The procedure can be applied to spatially inhomogeneous states [34]. Let us rewrite the Poisson bracket as

$$\{a, b\} = \frac{\partial a}{\partial J} \frac{\partial b}{\partial \theta} - \frac{\partial a}{\partial \theta} \frac{\partial b}{\partial J} \quad (\text{B10})$$

in terms of the angle-action coordinates. The linearized Vlasov equation can be written also in terms of the angle-action coordinates as

$$\frac{\partial f_1}{\partial t} + \Omega(J) \frac{\partial f_1}{\partial \theta} - f'_0(J) \frac{\partial}{\partial \theta} \mathcal{V}[f_1] = 0, \quad (\text{B11})$$

where  $\Omega(J) = d\mathcal{E}(J)/dJ$ . We omit to put the tilde over  $f_0$  and  $f_1$  to specify that the arguments of these functions are the angle-action variables in this section. We expand the functions  $f_1(\theta, J, t)$ ,  $\cos q(\theta, J)$ , and  $\sin q(\theta, J)$  into the Fourier series

$$f_1(\theta, J, t) = \sum_{k \in \mathbb{Z}} \hat{f}_{1,k}(J, t) e^{ik\theta}, \quad (\text{B12})$$

$$\cos q(\theta, J) = \sum_{k \in \mathbb{Z}} C^k(J) e^{ik\theta}, \quad (\text{B13})$$

$$\sin q(\theta, J) = \sum_{k \in \mathbb{Z}} S^k(J) e^{ik\theta}, \quad (\text{B14})$$

respectively. By using (B12), (B13), and (B14), the potential term  $\mathcal{V}[f_1]$  in (B11) is rewritten as

$$\begin{aligned} & -\mathcal{V}[f_1](q(\theta, J)) \\ &= \iint_{\mu} \cos(q(\theta, J) - q'(\theta', J')) f_1(\theta', J', t) d\theta' dJ' \\ &= 2\pi \sum_{m \in \mathbb{Z}} C^m(J) e^{im\theta} \sum_{k \in \mathbb{Z}} \int_L C^k(J')^* \hat{f}_{1,k}(J', t) dJ' \\ & \quad + 2\pi \sum_{m \in \mathbb{Z}} S^m(J) e^{im\theta} \sum_{k \in \mathbb{Z}} \int_L S^k(J')^* \hat{f}_{1,k}(J', t) dJ'. \end{aligned} \quad (\text{B15})$$

Then, the  $m$ th Fourier mode  $\hat{f}_{1,m}$  is shown to satisfy the equation

$$\begin{aligned} \frac{\partial \hat{f}_{1,m}}{\partial t} &= -im\Omega(J) \hat{f}_{1,m}(J, t) - 2\pi im C^m(J) f'_0(J) \\ & \quad \times \sum_{k \in \mathbb{Z}} \int_L C^k(J')^* \hat{f}_{1,k}(J', t) dJ' - 2\pi im S^m(J) f'_0(J) \\ & \quad \times \sum_{k \in \mathbb{Z}} \int_L S^k(J')^* \hat{f}_{1,k}(J', t) dJ'. \end{aligned} \quad (\text{B16})$$

Let  $\hat{f}_1^\lambda(\theta, J, t) = \sum_{m \in \mathbb{Z}} \hat{f}_{1,m}^\lambda(J, t) e^{im\theta}$  be an eigenfunction associated with an eigenvalue  $\lambda$  of  $\hat{\mathcal{L}}$ , i.e.,  $(\hat{f}_{1,m}^\lambda)_{m \in \mathbb{Z}}$  be the eigenvector associated with the eigenvalue  $\lambda$ . Setting

$$\hat{f}_{1,m}^\lambda(J, t) = \tilde{f}_{1,m}^\lambda(J) e^{\lambda t}, \quad \forall m \in \mathbb{Z} \quad (\text{B17})$$

and substituting it into (B16), we get

$$\begin{aligned} \tilde{f}_{1,m}^\lambda(J) = & -2\pi \frac{mf_0'(J)}{m\Omega(J) - i\lambda} C^m(J) \sum_{k \in \mathbb{Z}} \int_L C^k(J')^* \tilde{f}_{1,k}^\lambda(J') dJ' \\ & - 2\pi \frac{mf_0'(J)}{m\Omega(J) - i\lambda} S^m(J) \sum_{k \in \mathbb{Z}} \int_L S^k(J')^* \tilde{f}_{1,k}^\lambda(J') dJ'. \end{aligned} \quad (\text{B18})$$

Multiplying  $C^m(J)^*$  or  $S^m(J)^*$  to both sides of (B18), summing up over  $m \in \mathbb{Z}$ , and using the fact [34]

$$2\pi \sum_{m \in \mathbb{Z}} \int_L \frac{mf_0'(J)}{m\Omega(J) - i\lambda} C^m(J)^* S^m(J) dJ = 0, \quad (\text{B19})$$

we obtain the equations

$$\begin{aligned} & \left[ 1 + 2\pi \sum_{m \in \mathbb{Z}} \int_L \frac{mf_0'(J)}{m\Omega(J) - i\lambda} |C^m(J)|^2 dJ \right] \\ & \times \sum_{k \in \mathbb{Z}} \int_L C^k(J')^* \tilde{f}_{1,k}^\lambda(J') dJ' = 0 \end{aligned} \quad (\text{B20})$$

and

$$\begin{aligned} & \left[ 1 + 2\pi \sum_{m \in \mathbb{Z}} \int_L \frac{mf_0'(J)}{m\Omega(J) - i\lambda} |S^m(J)|^2 dJ \right] \\ & \times \sum_{k \in \mathbb{Z}} \int_L S^k(J')^* \tilde{f}_{1,k}^\lambda(J') dJ' = 0. \end{aligned} \quad (\text{B21})$$

A necessary condition for the existence of the nonzero eigenfunction corresponding to the eigenvalue  $\lambda$  is that at least one of the following two equations is satisfied:

$$\Lambda_x(i\lambda) \equiv 1 + 2\pi \sum_{m \in \mathbb{Z}} \int_L \frac{mf_0'(J)}{m\Omega(J) - i\lambda} |C^m(J)|^2 dJ = 0, \quad (\text{B22})$$

$$\Lambda_y(i\lambda) \equiv 1 + 2\pi \sum_{m \in \mathbb{Z}} \int_L \frac{mf_0'(J)}{m\Omega(J) - i\lambda} |S^m(J)|^2 dJ = 0.$$

When  $\text{Im}\omega > 0$ , the spectral functions  $\Lambda_x(\omega)$  and  $\Lambda_y(\omega)$  defined in (B22) coincide with the dispersion functions  $D_x(\omega)$  and  $D_y(\omega)$  defined in (26) and (36), respectively. As in the homogeneous state case, both  $\Lambda_x(\omega)$  and  $\Lambda_y(\omega)$  satisfy the relation (B9) since  $|C^m(J)| = |C^{-m}(J)|$  and  $|S^m(J)| = |S^{-m}(J)|$  are satisfied for all  $m \in \mathbb{Z}$ . It turns out that if  $\omega$  with  $\text{Im}\omega > 0$  is a root of the dispersion relation  $D_x(\omega) = 0$  or  $D_y(\omega) = 0$ , the linearized Vlasov operator has eigenvalues  $i\omega$ ,  $-i\omega$ ,  $-i\omega^*$ , and  $i\omega^*$ . We hence conclude that if  $\hat{\mathcal{L}}$  has an eigenvalue,  $\hat{\mathcal{L}}$  has inevitably an unstable eigenvalue, so that the stationary solution  $f_0(J)$  should be unstable.

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